ON THE METHOD OF ORTHOGONAL POLYNOMIALS IN PLANE MIXED PROBLEMS OF ELASTICITY THEORY

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Besides asymptotic methods [1], the method of orthogonal polynomials [2-12] has become quite widespread in recent years in investigations of complex mixed problems of elasticity theory. Its essence is the following. The mixed problem is reduced to the solution of an integral equation of the first kind, some domain of variation of the dimensionless parameters in the kernel of the integral equation is considered, and the principal (singular) part of the kernel which correponds to the selected domain of parameter variation is isolated. Eigenfunctions of the integral operator corresponding to the principal part of the kernel are found, where in the majority of cases known at present some system of classical orthogonal polynomials turns out to consist of eigenfunctions. The known function in the right side of the integral equation, and the solution are expanded in a series in these polynomials. The regular part of the kernel is expanded in a double series. Afterwards, the integral equation is easily reduced to an infinite algebraic system. Under appropriate truncation (*) of the infinite system, the matrix of the final system obtained turns out to be almost triangular, which permits sufficiently easy numerical solution of the problem.

A foundation for the method of orthogonal polynomials, and a numerical example for the integral equation

$$\int_{-1}^{1} \varphi(\xi) K\left(\frac{\xi-x}{\lambda}\right) d\xi = \pi f(x) \quad (|x| \leqslant 1), \qquad K(t) = \int_0^\infty \frac{L(u)}{u} \cos ut\, du \tag{0.1}$$

$$|1 - L(u)| \leqslant e^{-\nu u} \sum_{i=0}^{s} B_i u^i \quad (0 < u < \infty), \qquad L(u) \sim Au \quad (u \to 0) \tag{0.2}$$

are given in Sects. 1 and 2. Here ν, B_i, A are positive constants, and the function $L(u)$ is bounded for $0 < u < \infty$. The scheme of the orthogonal polynomials method for equations of the type (0.1), (0.2) is substantially elucidated in [2, 4, 5, 9, 12].

The scheme of the orthogonal polynomials method for the integral equation

$$-\int_k^1 \varphi(\xi) \ln \frac{|\xi^2 - x^2|}{\lambda^2} d\xi = \pi f(x) + \int_k^1 \varphi(\xi) G\left(\frac{\xi}{\lambda}, \frac{x}{\lambda}, \frac{k}{\lambda}\right) d\xi \quad (k \leqslant x \leqslant 1) \tag{0.3}$$

is presented in Sect. 3.

Here $\lambda \in (0, \infty)$ and $k \in (0, 1)$ are dimensionless parameters, G is a sufficiently smooth function and symmetric relative to ξ, x. Equations of type (0.3) originate in investigations of plane, mixed problems, even in x, and with two contact sections, in elasticity theory.

1. Let us note that a kernel $K(t)$ of the form (0.1) can be represented as [1]

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*) See below for details.

$$K(t) = -\ln|t| - F(t), \quad F(t) = \int_0^\infty \frac{[1-L(u)]\cos ut - e^{-u}}{u}\,du \tag{1.1}$$

Utilizing (0.2), it can be said that the function $F(t)$ is continuous with all its derivatives in $0 \leqslant |t| < \infty$.

On the basis of (1.1) let us rewrite the integral equation (0.1) as

$$-\int_{-1}^{1} \varphi_*(\xi) \ln\frac{|\xi - x|}{\lambda}\,d\xi = \pi f_*(x) + \int_{-1}^{1} \varphi_*(\xi) F\left(\frac{\xi - x}{\lambda}\right) d\xi \quad (|x| \leqslant 1) \tag{1.2}$$

$$f_*(x) = \frac{1}{\pi}\int_{-1}^{1} \varphi_0(\xi)\left[F\left(\frac{\xi - x}{\lambda}\right) - F(0)\right] d\xi, \quad \varphi(\xi) = \varphi_0(\xi) + \varphi_*(\xi) \tag{1.3}$$

where $\varphi_0(\xi)$ is determined from the equation

$$-\int_{-1}^{1} \varphi_0(\xi) \ln\frac{|\xi - x|}{\lambda}\,d\xi = \pi f(x) + F(0) P_0 \quad (|x| \leqslant 1), \qquad P_0 = \int_{-1}^{1} \varphi_0(\xi)\,d\xi \tag{1.4}$$

Theorem 1.1. If $f'(x) \in L_{(4/3+0)}(-1,\ 1)$, the solution of the integral equation (1.4) exists, is unique, belongs to $L_{(4/3-0)}(-1,\ 1)$ and has the form

$$\varphi_0(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[P_0 - \int_{-1}^{1} \frac{f'(t)\sqrt{1-t^2}}{t-x}\,dt\right], \quad P_0 = \frac{1}{\ln 2\lambda - F(0)} \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} \tag{1.5}$$

Moreover, if the solution of the integral equation (1.2), (1.3) exists in $L_1(-1,\ 1)$ for $\lambda \in (0,\ \infty)$, it has the form

$$\varphi_*(x) = \Phi_*(x)(1 - x^2)^{-1/2} \tag{1.6}$$

where the function $\Phi_*(x)$ is continuous with all its derivatives for $x \in [-1,\ 1]$.

The proof of this theorem is actually contained in [1, 5, 12, 13].

We shall assume that the conditions of Theorem 1.1 are satisfied, and the function $f(x)$ is even (the even case of the integral equation (0.1)). As has been shown in [14, 15], the solution for the odd case can be obtained from a special even solution by differentiation.

Let us seek the function $\Phi_*(x)$ in the relationship (1.6) in the form of the following series of Chebyshev polynomials:

$$\Phi_*(x) = \sum_{k=0}^{\infty} S_k T_{2k}(x) \qquad (|x| \leqslant 1) \tag{1.7}$$

The series (1.7) converges uniformly [16] by virtue of the properties of the function $\Phi_*(x)$ mentioned in Theorem 1.1.

Let us also expand a function $f_*(x)$ of the form (1.3) into the series

$$f_*(x) = \sum_{k=0}^{\infty} R_{k*} T_{2k}(x) \qquad (|x| \leqslant 1) \tag{1.8}$$

On the basis of the above-mentioned properties of the functions $\varphi_0(x)$ and $F(t)$ it is not difficult to show that the function $f_*(x)$ is continuous with its derivatives. Hence, the series (1.8) also converges uniformly to $f_*(x)$.

Finally, let us expand the function $F(t)$ $(0 \leqslant |t| < \infty)$ in a double series of

Chebyshev polynomials. We have

$$F\left(\frac{\xi-x}{\lambda}\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[C_{mn}(\lambda)\,T_{2m}(x)\,T_{2n}(\xi)+\ldots\right] \tag{1.9}$$

Terms containing Chebyshev polynomials with odd indices are not written down in the relationships (1.9); they will not be needed henceforth.

Utilizing the known orthogonality property of Chebyshev polynomials, we obtain

$$C_{mn}(\lambda)=\frac{\beta_{mn}}{\pi^2}\int_0^{\pi}\int_0^{\pi}F\left(\frac{\cos\psi-\cos\varphi}{\lambda}\right)\cos 2m\varphi\cos 2n\psi\, d\varphi d\psi \tag{1.10}$$

$$\beta_{00}=1,\quad \beta_{m0}=\beta_{0n}=2,\quad \beta_{mn}=4$$

Lemma 1.1. For all values $0\leqslant |t|=|\xi-x|\lambda^{-1}\leqslant 2\lambda^{-1}<\infty$ the series (1.9) converges uniformly to $F(t)$ in the set of variables ξ, x. The following estimates hold for the coefficients of the series $C_{mn}(\lambda)$:

$$|C_{mn}(\lambda)|\leqslant\beta_{mn}\max|F(t)| \tag{1.11}$$

$$|C_{mn}(\lambda)|\leqslant d\lambda^{-3}[(n^2-1)n]^{-1}\qquad (d=\text{const},\ n\geqslant 2) \tag{1.12}$$

$$C_{mn}(\lambda)|\leqslant D\lambda^{-6}[(m^2-1)(n^2-1)mn]^{-1}\qquad (D=\text{const},\ m\geqslant 2,\ n\geqslant 2) \tag{1.13}$$

$$|C_{mn}(\lambda)|\leqslant\frac{\beta_{mn}}{(2m)!\,(2n)!}\left(\frac{1}{2\lambda\nu}\right)^{2m+2n}\sum_{i=0}^{s}B_i\frac{(2m+2n+i-1)!}{\nu^i}\qquad (m+n\geqslant 1) \tag{1.14}$$

$$C_{mn}(\lambda)\sim 0,\quad (m\neq n),\qquad C_{00}(\lambda)\sim\ln 2\lambda$$

$$C_{mm}(\lambda)\sim m^{-1}\quad (m\geqslant 1)\quad \text{for}\quad \lambda\to 0 \tag{1.15}$$

The estimate (1.11) follows at once from (1.10). The estimates (1.12) and (1.13) are obtained from (1.10) by integration by parts. Uniform convergence of the series (1.9) results from the estimate (1.13).

To prove the estimates (1.14), (1.15), let us substitute the expression (1.1) for the function $F(t)$ into (1.10). Integrating, we represent the coefficients $C_{mn}(\lambda)$ as

$$C_{00}(\lambda)=\int_0^{\infty}\frac{[1-L(u)]\,J_0^2(u/\lambda)-e^{-u}}{u}\,du\qquad (J_k(x)\ \text{is the Bessel function})$$

$$C_{mn}(\lambda)=(-1)^{m+n}\beta_{mn}\int_0^{\infty}[1-L(u)]\,J_{2m}\left(\frac{u}{\lambda}\right)J_{2n}\left(\frac{u}{\lambda}\right)\frac{du}{u}\qquad (m+n\geqslant 1) \tag{1.16}$$

On the basis of (1.16), taking account of the estimate (0.2) and

$$|J_{2s}(x)|\leqslant\frac{1}{(2s)!}\left(\frac{x}{2}\right)^{2s}\qquad (x>0,\ s\geqslant 0) \tag{1.17}$$

we obtain (1.15) and (1.16).

Let us note that for $\lambda>2/\nu$ uniformly convergent expansions can also be obtained for the coefficients $C_{mn}(\lambda)$.

An estimate of type (1.12) holds for the coefficients R_{k*} of the series (1.8), namely:

$$|R_{k*}|\leqslant C[(k^2-1)k]^{-1}\qquad (C=\text{const},\ k\geqslant 2) \tag{1.18}$$

Moreover, the following formulas are valid [12]:

$$R_{k*} = C_{k0}(\lambda) R_0 [\ln 2\lambda - F(0)]^{-1} + \sum_{n=1}^{\infty} nC_{kn}(\lambda) R_n$$

$$R_{0*} = R_0 [C_{00}(\lambda) - F(0)] [\ln 2\lambda - F(0)]^{-1} + \sum_{n=1}^{\infty} nC_{0n}(\lambda) R_n \qquad (1.19)$$

$$\frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_{2n}(x)}{\sqrt{1-x^2}} dx = \begin{cases} 2R_0 \, (n=0) \\ R_n \, (n \neq 0) \end{cases}$$

We finally obtain relationships to determine the coefficients S_k in (1.7).

Theorem 1.2. A sequence of numbers S_k belonging to l_1 and satisfying infinite system of linear algebraic equations

$$R_{k*} + S_0 C_{k0}(\lambda) + \frac{1}{2} \sum_{n=1}^{\infty} S_n C_{kn}(\lambda) = \begin{cases} S_0 \ln 2\lambda & (k=0) \\ S_k (2k)^{-1} & (k=1,2,\ldots) \end{cases} \qquad (1.20)$$

corresponds to any solution $\varphi_*(x)$ of the integral equation (1.2) from the class $L_1(-1, 1)$, and vice versa.

For the proof, let us substitute the functions $\varphi_*(\xi)$, $f_*(x)$ and $F(t)$ in the form (1.6)–(1.9) into the integral equation (1.2), and let us evaluate the integrals by utilizing formula (2.7) from [5], and the orthogonality property of Chebyshev polynomials. We obtain a relationship in whose right and left sides there are series in even Chebyshev polynomials. Equating coefficients on both sides of polynomials of the same number, we obtain the infinite system (1.20).

Taking account of the fact that the function $\Phi_*(x)$ is continuous with all its derivatives, we can prove that the sequence of numbers S_k belongs to the class l_p, $p \geqslant 1$. In order to see this, it is sufficient to obtain an estimate of the type (1.8) for the numbers S_k.

Now, let the sequence S_k be the solution of the infinite system (1.20) and let it belong to l_1. Then the series (1.7) converges uniformly to some continuous function $\Phi_*(x)$ for $x \in [-1,1]$. By an inverse transformation of the infinite system (1.20), into the integral equation (1.2) it is easy to see that the function $\varphi_*(x) = \Phi_*(x)(1-x^2)^{-1/2}$, which belongs to $L_1(-1,1)$, is a solution of (1.2).

2. Let us rewrite the system (1.20) in the more convenient form

$$S_0 (\ln 2\lambda - C_{00}) = R_{0*} + \frac{1}{2} \sum_{n=1}^{\infty} S_n C_{0n} \qquad (2.1)$$

$$x_i = \sum_{k=1}^{\infty} a_{ik} x_k + b_i \qquad (i=1,2,\ldots) \qquad (2.2)$$

$$x_i = S_i (2i)^{-1}, \quad a_{ik} = kC_{ik}, \quad b_i = R_{i*} + S_0 C_{i0}$$

Let us note that

$$P = \int_{-1}^{1} \varphi(\xi) d\xi = \int_{-1}^{1} [\varphi_0(\xi) + \varphi_*(\xi)] d\xi = P_0 + P_* \qquad (2.3)$$

where P_0 has the form (1.5) and P_* is given, by virtue of the relationships (1.6), (1.7), by the simple formula

$$P_* = \pi S_0 \tag{2.4}$$

By assumption $\varphi_*(x) \in L_1(-1, 1)$, therefore $|S_0| < \infty$.

Solving the infinite system (2.2), we then find S_0 from (2.1).

Theorem 2.1. The infinite system (2.2) is quasi-completely regular for $\lambda > 0$. If its bounded solution exists, the sequence x_i belongs to l_p, $p \geqslant 1$. As $\lambda \to 0$ the determinant of the system tends to zero.

As is known [17], an infinite system will be quasi-completely regular, if

$$A_i = \sum_{k=1}^{\infty} |a_{ik}| < \infty \qquad (i=1,2,\ldots N)$$

$$A_i \leqslant 1 - \theta < 1 \qquad (i=N+1, N+2,\ldots) \tag{2.5}$$

$$b_i \leqslant K\theta \qquad (i=N+1, N+2,\ldots)$$

The compliance with conditions (2.5) for $\lambda > 0$ follows easily from the estimates (1.11) – (1.13), (1.18).

Now, let a bounded solution of the system (2.2) be found

$$|x_i| \leqslant X \qquad (X=\mathrm{const}) \tag{2.6}$$

We then have

$$|x_i| \leqslant X \sum_{k=1}^{\infty} |a_{ik}| + |b_i| \tag{2.7}$$

or on the basis of the estimates (1.12), (1.13) and (1.18)

$$|x_i| \leqslant x^* [(i^2 - 1)\, i]^{-1} \qquad (x^*=\mathrm{const},\ i>2) \tag{2.8}$$

It hence follows that $\{x_i\} \in l_p$, $p \geqslant 1$.

It follows at once from the estimate (1.15) that the determinant of the system (2.2) tends to zero as $\lambda \to 0$.

On the basis of the theorem proved it can be concluded that the existence and uniqueness of the solution of the infinite system (2.2) for $\lambda > 0$ reduces to the existence of a solution of an infinite system of the first N equations. For $\lambda \lll 1$ the matrix of the system (2.2) becomes poorly specified. Under the condition (2.6) the series in (2.1) converges absolutely on the basis of the estimates (2.8) and (1.12).

Theorem 2.2. For $\lambda > \lambda_0$ the infinite system (2.2) is completely regular.

For the proof we estimate the quantities A_i $(i = 1, 2, \ldots)$. On the basis of the estimate (1.14) we have

$$A_i \leqslant 2 \sum_{j=0}^{s} \frac{B_j}{(2i)!\,\nu^j} \sum_{k=1}^{\infty} \frac{(2i+2k+j-1)!}{(2k-1)!} \left(\frac{1}{2\lambda\nu}\right)^{2i+2k} \tag{2.9}$$

By mathematical induction it can be proved that

$$\sum_{n=0}^{\infty} \frac{(2n+r)!}{(2n+1)!} q^{2n} = \frac{(r-1)!}{(1-q^2)^r} \sum_{k=0}^{m} \binom{r}{2k+1} q^{2k} \leqslant \frac{(2r-2)!!}{(1-q^2)^r} \tag{2.10}$$
$$(0 \leqslant q < 1)$$

where m is the integer part of the number $(r-1)/2$. Taking account of (2.10) the estimate (2.19) becomes

$$A_i \leqslant 2 \sum_{j=0}^{s} B_j \frac{(4i+2j)!!\, q^{2i+2}}{\nu^j (2i)!\, (1-q^2)^{2i+j+1}} = A_i^* \quad \left(q = \frac{1}{2\lambda\nu}\right) \tag{2.11}$$

It hence is seen that all $A_i < \infty$, provided that $q < 1$, $\lambda > {}^1/_2\nu$.

Now, let us find the condition for which $A_{i+1}^* < A_i^*$ $(i \geqslant 1)$. On the basis of (2.11) we have

$$D_{ij}(q) = \frac{4q^2(2i+j+1)(2i+j+2)}{(1-q^2)^2(2i+1)(2i+2)} < 1 \tag{2.12}$$

Let us note that for any fixed $q < 1$ and $j \geqslant 1$ the numbers $D_{ij}(q)$ decrease monotonously as i increases, and for $j = 0$ the numbers $D_{i0}(q) = 4q^2(1-q^2)^{-2}$.

Therefore, the inequality (2.12) will be satisfied for all i and j, if

$$D_{0s}(q) = 3^{-1}q^2(3+s)(4+s)(1-q^2)^{-2} < 1 \tag{2.13}$$

We hence find the greatest q_1 and its corresponding λ_1.

Let us now clarify when $A_1^* < 1$. On the basis of (2.11) we have

$$\sum_{j=0}^{s} B_j \frac{(2j+4)!!\,q^4}{\nu^4(1-q^2)^{j+3}} < 1 \tag{2.14}$$

We hence find the greatest q_2 and its corresponding λ_2.

From all the above there results that the system (2.2) is completely regular for

$$\lambda > \lambda_0 = \sup({}^1/_2\nu, \lambda_1, \lambda_2) \tag{2.15}$$

It follows from the theorem proved that for $\lambda > \lambda_0$ the infinite system (2.2) has a unique bounded solution, and therefore, a solution in l_p, $p \geqslant 1$ (see Theorem 2.1), which can be obtained by successive approximations.

As an illustration, let us examine the problem of impression of a stamp on an elastic strip lying on a rigid foundation. There are no friction forces between the stamp and the strip, as well as between the strip and the foundation.

As is known [18], this problem can be reduced to the solution of an integral equation of the form (0.1). The function $L(u)$ is given by the second formula in (1.14) in [18]. It is easy to see that it satisfies conditions (0.2), where

$$s = 1, \quad \nu = 2, \quad B_0 = 2, \quad B_1 = 4 \tag{2.16}$$

In conformity with (2.16), we find by the scheme described above that the solution of the mentioned problem exists and is unique in $L_1(-1,1)$ if $f'(x) \in L_{(4/_3+0)}(-1,1)$ and $\lambda > \lambda_0 = 0.883$.

Let us utilize the method of reduction to find the approximate solutions of the infinite system (2.2).

Let us note that any truncation of the series (1.9) for $F(t)$ automatically results in a corresponding truncation of the infinite system (2.2). Hence, it is natural to truncate the series (1.9) in such a manner as to obtain, as a result, a solution of the infinite system convenient for practice.

Proceeding from the above, let us retain only terms for which $m + n \leqslant r$ in the series (1.9). It is then easy to see that the finite system of linear algebraic equations obtained is of the form

$$x_i = \sum_{k=1}^{r-i} a_{ik}x_k + b_i^r, \quad b_i^r = R_{i*}^r + S_0C_{i0} \quad (i = 1, 2, \ldots r) \tag{2.17}$$

The relationship (2.1) also changes form somewhat

$$S_0(\ln 2\lambda - C_{00}) = R_{0*}^r + \frac{1}{2}\sum_{k=1}^{r} S_kC_{0k} \tag{2.18}$$

The superscript r in the quantity R_{i*}^r means that the summation over n must be made up to r in (1.19).

The rather unusual truncation of the infinite system (2.2) described, results in a question on the convergence of the method of reduction.

Lemma 2.1. Let be given a sequence of completely regular infinite systems

$$x_i = \sum_{k=1}^{\infty} a_{ik}^r x_k + b_i^r, \qquad |b_i^r| \leqslant C_r \tag{2.19}$$

as well as the completely regular infinite system

$$x_i = \sum_{k=1}^{\infty} a_{ik} x_k + b_i, \qquad |b_i| \leqslant C \tag{2.20}$$

Moreover, let

$$\lim a_{ik}^r = a_{ik}, \quad \lim b_i^r = b_i \quad \text{for} \quad r \to \infty \tag{2.21}$$

Then if x_i^r is a solution of the system (2.19), and x_i of the system (2.20), then

$$\lim x_i^r = x_i \quad \text{for} \quad r \to \infty \tag{2.22}$$

The lemma presented is a particular case of Theorem III (Ch. I, Sect. 2 in [17]).

It follows from the lemma that the method of reduction expounded above converges, i.e. the solution x_i^r of the truncated system (2.17) tends to a solution of the infinite system (2.2) as $r \to \infty$ if $\lambda > \lambda_0$. For $\lambda_0 > \lambda > 0$ the method of reduction evidently also converges if the following finite system is solvable:

$$x_i = \sum_{k=1}^{\infty} a_{ik} x_k + b_i \quad (i=1,2,\ldots N) \tag{2.23}$$

A practical solution of the truncated system (2.17) is produced simply enough because its coefficients form an almost triangular matrix. After the quantities x_i have been determined from the system (2.17), we find S_0 from the relationship (2.18), and then the approximate solution of the integral equation (0.1) by means of (1.3), (1.5)−(1.7), (2.2). The summation over k in (1.7) is carried out up to r.

Let us note that for a given accuracy of the approximate solution of the integral equation (0.1) and a decrease in the parameter λ the number of equations in the truncated system (2.17) must be increased. This follows from the fact that the series (1.9), as can be shown on the basis of (1.15), will diverge on the line $\xi = x$ as $\lambda \to 0$.

It should be noted that if the function $f(x)$ is sufficiently smooth, it is expedient to separate the solution $\varphi(\xi)$ into the components $\varphi_0(\xi)$ and $\varphi_*(\xi)$; in this case the method expounded above must be applied directly to the integral equation

$$-\int_{-1}^{1} \varphi(\xi) \ln \frac{|\xi - x|}{\lambda}\, d\xi = \pi f(x) + \int_{-1}^{1} \varphi(\xi) F\left(\frac{\xi - x}{\lambda}\right) d\xi \qquad (|x| \leqslant 1) \tag{2.24}$$

As an illustration, the above-mentioned problem of impressing a flat stamp ($f(x) \equiv 1$) on an elastic strip was considered. The coefficients $C_{mn}(\lambda)$ of the series (1.9) were computed on an electronic computer and are presented in Table 1.

The approximate solutions of equation (0.1) are finally represented as

$$\varphi(x) = (1 - x^2)^{-1/2} \sum_{i=0}^{r} d_i x^{2i}, \quad \psi = \lim \varphi(x)(1 - x^2)^{1/2} \quad \text{for} \quad x \to 1 \tag{2.25}$$

The coefficients d_i are given in Table 2.

Some characteristics of the solution (2.25) with appropriate data obtained by asymptotic methods in [13, 14, 19] are compared in Table 3.

Table 1

mn	$\lambda = 2$	$\lambda = 1$	$\lambda = 1/2$	$\lambda = 1/4$	mn	$\lambda = 1/4$
00	-0.2376	$-0.1064 \cdot 10^{-1}$	0.3946	0.9180	32	$0.4322 \cdot 10^{-1}$
10	$0.5139 \cdot 10^{-1}$	0.1204	0.1556	0.1317	41	$-0.2954 \cdot 10^{-2}$
11	$-0.9356 \cdot 10^{-2}$	$-0.7736 \cdot 10^{-1}$	-0.2848	-0.5268	50	$-0.2633 \cdot 10^{-3}$
20	$-0.6062 \cdot 10^{-3}$	$-0.1965 \cdot 10^{-2}$	$0.1047 \cdot 10^{-1}$	$0.3546 \cdot 10^{-1}$	33	$-0.4656 \cdot 10^{-1}$
21		$0.7912 \cdot 10^{-2}$	$0.5436 \cdot 10^{-1}$	0.1037	42	$-0.1415 \cdot 10^{-2}$
30		$-0.7350 \cdot 10^{-4}$	$-0.4012 \cdot 10^{-2}$	$0.2613 \cdot 10^{-2}$	51	$-0.6880 \cdot 10^{-3}$
22			$-0.1610 \cdot 10^{-1}$	-0.1412	60	$0.1320 \cdot 10^{-4}$
31			$-0.3780 \cdot 10^{-2}$	$0.1055 \cdot 10^{-1}$		
40			$\sim 10^{-8}$	$-0.1460 \cdot 10^{-2}$		

Table 2

λ	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
2	0.9652	-0.198	0.0169				
1	1.909	-0.925	0.139	0.00352			
$1/2$	3.937	-2.435	-2.640	1.843	$\sim 10^{-4}$		
$1/4$	8.00	-4.01	-1.80	2.03	-7.47	7.18	-1.72

Table 3

λ	2	2[13]	2[19]	1	1[19]	$1/2$	$1/2$[19]	$1/4$	$1/4$[14]
$\varphi(0)$	0.965	0.967	0.968	1.909	1.92	3.937	3.97	8.000	7.991
$\varphi(1/2)$	1.059	1.074	1.05	1.948	1.95	3.955	3.92	7.978	7.941
ψ	0.784	0.809	0.790	1.126	1.12	1.638	1.58	2.263	2.257
P	2.742	2.800	2.75	4.712	4.70	8.70	8.71	16.71	16.71

3. Let us turn to an examination of the integral equation (0.3); let us make a change of variable therein, and introduce the following notation

$$x = \sqrt{k'^2 u^2 + k^2}, \quad \xi = \sqrt{k'^2 v^2 + k^2}, \quad k' = \sqrt{1 - k^2}$$

$$\varphi_+(v) = k'^2 v (k'^2 v^2 + k^2)^{-1/2} \varphi\left(\sqrt{k'^2 v^2 + k^2}\right), \quad f_+(u) = f\left(\sqrt{k'^2 u^2 + k^2}\right) \tag{3.1}$$

$$G_+\left(\frac{v}{\mu}, \frac{u}{\mu}, \varepsilon\right) = G\left(\sqrt{\mu^{-2} v^2 + \varepsilon^2}, \sqrt{\mu^{-2} u^2 + \varepsilon^2}, \varepsilon\right) \quad \varepsilon = \frac{k}{\lambda}, \quad \mu = \frac{\lambda}{k'}$$

We have

$$-\int_0^1 \varphi_+(v) \ln \frac{|v^2-u^2|}{\mu^2}\, dv = \pi f_+(u) + \int_0^1 \varphi_+(v)\, G_+\left(\frac{v}{\mu}, \frac{u}{\mu}, \varepsilon\right) dv \quad (0 \leqslant u \leqslant 1) \tag{3.2}$$

Let us note that Eq. (3.2) is substantially an even modification of an integral equation of the type (2.24). We seek the solution of (3.2) in the form

$$\varphi_+(v) = \varphi_0(v) + \varphi_1(v) \tag{3.3}$$

where $\varphi_0(v)$ is determined from (1.4) and has the form (1.5) if x, λ, $f(x)$ and $F(0)$ in the formulas mentioned are replaced by u, μ, $f_+(u)$ and $G_+(0)$, respectively. The function $\varphi_1(v)$ is then found from (3.2) in which it is necessary to take

$$f_1(u) = \frac{1}{\pi}\int_0^1 \varphi_0(v)\left[2G_+(0) - G_+\left(\frac{v}{\mu}, \frac{u}{\mu}, \varepsilon\right)\right] dv \tag{3.4}$$

in place of $f_+(u)$.

We seek $\varphi_1(v)$ in the form

$$\varphi_1(v) = \Phi_1(v)(1-v^2)^{-1/2}, \qquad \Phi_1(v) = \sum_{i=0}^{\infty} q_i T_{2i}(v) \tag{3.5}$$

Let us expand the functions $G_+(v/\mu, u/\mu, \varepsilon)$ and $f_1(u)$ into the series

$$G_+\left(\frac{v}{\mu}, \frac{u}{\mu}, \varepsilon\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e_{mn}(\mu, \varepsilon)\, T_{2m}(u)\, T_{2n}(v), \qquad f_1(u) = \sum_{i=0}^{\infty} R_{i1} T_{2i}(u) \tag{3.6}$$

Formulas analogous to (1.10) and (1.19) hold for the coefficients $e_{mn}(\mu, \varepsilon)$ and R_{i1}

Substituting (3.5), (3.6) into the integral equation for $\varphi_1(v)$ and performing all the required operations, we arrive at an infinite system in q_i

$$R_{i1} + q_0 e_{i0}(\mu, \varepsilon) + \frac{1}{2}\sum_{n=1}^{\infty} q_n e_{in}(\mu, \varepsilon) = \begin{cases} q_0 \ln 2\mu & (i=0) \\ q_i (2i)^{-1} & (i = 1, 2, \ldots) \end{cases} \tag{3.7}$$

Just as for (1.20), the approximate solution of the system (3.7) can be found by the method of reduction described in Sect. 2.

Under appropriate constraints imposed on the function $G(\xi/\lambda, x/\lambda, k/\lambda)$ a foundation can be supplied for the method of orthogonal polynomials for equation (0.3) just as has been done for (0.1) above.

Let us note that the scheme elucidated in [12] for the method of orthogonal polynomials for (1.3) can also be given a vigorous foundation.

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BIBLIOGRAPHY

1. Aleksandrov, V. M., Asymptotic methods in contact problems of elasticity theory. PMM Vol. 32, №4, 1968.
2. Popov, G. Ia., On an approximate method of solving some plane contact problems of elasticity theory. Izv. Akad. Nauk ArmSSR, Fiz.-Matem. Nauk, Vol. 14, №3, 1961.
3. Popov, G. Ia., The contact problem of the theory of elasticity for the case of a circular area of contact. PMM Vol. 26, №1, 1962.

4. Popov, G. Ia, Some properties of classical polynomials and their application to contact problems. PMM Vol. 27, №5, 1963.
5. Aleksandrov, V. M., On an approximate solution of a class of integral equations. Izv. Akad. Nauk ArmSSR, Fiz.-Matem. Nauk, Vol. 17, №2, 1964.
6. Popov, G. Ia., Certain properties of classical polynomials and their application in contact problems. PMM Vol. 28, №3, 1964.
7. Popov, G. Ia., On an approximate method of solving the integral equations of electromagnetic wave diffraction by a strip of finite width. Zh. Tekh. Fiz., Vol. 35, №3, 1965.
8. Popov, G. Ia., Plane contact problem of the theory of elasticity with bonding or frictional forces. PMM Vol. 30, №3, 1966.
9. Lutchenko, S. A., On the impression of a stamp on the side surface of an elastic foundation in the shape of a wedge. Prikl. Mekh., Vol. 2, №12, 1966.
10. Popov, G. Ia., Impression of a punch on a linearly-deforming foundation taking into account friction forces. PMM Vol. 31, №2, 1967.
11. Popov, G. Ia., On an approximate method of solving the contact problem of an annular stamp. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol. 20, №2, 1967.
12. Aleksandrov, V. M., On the approximate solution of some integral equations of the theory of elasticity and mathematical physics. PMM Vol. 31, №6, 1967.
13. Aleksandrov, V. M., On the approximate solution of a certain type of integral equations. PMM Vol. 26, №5, 1962.
14. Aleksandrov, V. M., Babeshko, V. A. and Kucherov, V. A., Contact problems for an elastic layer of slight thickness. PMM Vol. 30, №1, 1966.
15. Aleksandrov, V. M., On a contact problem for an elastic wedge. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol. 20, №1, 1967.
16. Natanson, I. P., Constructive Theory of Functions. Moscow-Leningrad, Gostekhteorizdat, 1949.
17. Kantorovich, L. V. and Krylov, V. I., Approximate Methods of Higher Analysis. Moscow, Gostekhteorizdat, 1950.
18. Aleksandrov, V. M. and Kucherov, V. A., Some problems on the effect of two stamps on an elastic strip. Inzh. Zh., Mekh. Tverd. Tela, №4, 1968.
19. Aleksandrov, V. M. and Babeshko, V. A., Contact problems for an elastic strip of small thickness. Izv. Akad. Nauk SSSR, Mekhanika, №2, 1965.

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